Spectral properties of stochastic electromagnetic fields with spherical symmetry

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We discuss the general solution of equations of the second-order correlation theory in the space-frequency domain for classical, statistically stationary vector electromagnetic fields. The spectral properties of radiation are studied in detail for fields which are spherically symmetric in the statistical sense. The formalism of cross-spectral tensors and mode decomposition are used. It is shown that the frequency dependence of the spectrum of radiation may differ in different points of space.

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I. INTRODUCTION

In the past few years there has been a good deal of research regarding the spectral correlation properties of partially coherent radiation. In particular, studies of changes in spectra of radiation induced by coherence properties of sources have become of considerable interest following a theoretical prediction that, in general, the normalized spectrum of light is not invariant on propagation, even in free space [1]. This prediction has been subsequently verified experimentally by several groups [2–4]. Numerous publications both theoretical [5–8] and experimental [9–11] dealing with this phenomenon have appeared since then. The changes in the spectrum which are dependent on the state of coherence of the source have been studied for planar quasihomogeneous sources in detail [1,6,12]. It has also been shown that the changes in the spectrum of radiation on propagation occur even when the stochastic scalar field is spherically symmetric [13–15].

The results demonstrate that this effect, often called the Wolf effect, is more than just a spatial redistribution of particular frequency components of the radiation. The effect stems rather from changes and an increase in the degree of coherence of radiation during propagation. It is worth mentioning that it was demonstrated in [14,15] that there is no violation of the law of conservation of energy in such situations.

Most investigations carried out so far have been performed on the basis of stochastic scalar wave theory. But there are situations where the vector nature of light cannot be neglected, for example, situations involving polarization properties of light, problems with a certain degree of symmetry, etc. Generalization from scalar to vector theory (or tensor theory—using correlation tensors) is not straightforward.

Although the basic equations of the second-order correlation theory of the electromagnetic fields have been formulated a long time ago, both in the classical [16] and in the quantum domain [17], and also within the framework of the space-time [16] and the space-frequency formulation [18], little is known about the solutions of these equations, except for blackbody radiation [19,20].

In the present paper we discuss the general solution of these equations in the space-frequency domain for classical fields which are spherically symmetric in the statistical sense (of course Maxwell's equations for a deterministic field do not have a spherically symmetric solution). More specifically, we study correlation properties of stationary stochastic radiative electromagnetic fields of spherical symmetry, employing the cross-spectral tensors of the second order and their expansions in terms of vector spherical harmonics.

II. STOCHASTIC ELECTROMAGNETIC FIELD AND CROSS-SPECTRAL TENSOR

We begin with a deterministic complex electric field $\mathbf{E}(\mathbf{r},t)$ in free space [21] whose components $E_j(\mathbf{r},t)$, $j = 1,2,3$, are the complex analytic signals of the components of the real electric strength vector and which satisfies (outside the sources) the homogeneous wave equation, the condition of transversity

$$\operatorname{div}\mathbf{E}(\mathbf{r},t) = 0, \tag{1}$$

and the radiation condition

$$\frac{\partial E_j(\mathbf{r},t)}{\partial r} + \frac{1}{c}\,\frac{\partial E_j(\mathbf{r},t)}{\partial t} = o(1/r) \quad \text{for } r = |\mathbf{r}| \to \infty. \tag{2}$$

We will also suppose the field decreases as $1/r$ for large r. Here $\mathbf{r}$ denotes a position vector of a point in space, t is an arbitrary instant of time, and c is the speed of light *in vacuo.* The symbol $o(x)$ denotes a quantity of higher order of smallness than x for $x \to 0$.

For stochastic, statistically stationary (at least in the wide sense) field, the (electric) correlation tensor can be defined by the expression

$$\Gamma_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},t^{(2)}-t^{(1)}) = \left\langle E_i^*(\mathbf{r}^{(1)},t^{(1)})E_j(\mathbf{r}^{(2)},t^{(2)})\right\rangle. \tag{3}$$

Here $\langle\rangle$ denotes the ensemble average and the asterisk denotes the complex conjugate. The cross-spectral tensor is given as the Fourier transform of correlation tensor:

$$W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\Gamma_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\tau)e^{-i\omega\tau}d\tau, \quad (4)$$

where ω is the frequency.

From the definition of cross-spectral tensor $W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$ and from the properties of complex electric strength $\mathbf{E}(\mathbf{r},t)$ the equations for the cross-spectral tensor can be easily derived. They are

$$\partial_m^{(\lambda)}\partial_m^{(\lambda)}W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+k^2W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=0 \quad \text{outside the sources}, \quad (5)$$

$$\partial_i^{(1)}W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=0,$$
$$\partial_j^{(2)}W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=0. \quad (6)$$

Subject to the asymptotic boundary conditions

$$\frac{\partial}{\partial r^{(1)}}W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)-ikW_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=o(1/r^{(1)}),$$
$$\frac{\partial}{\partial r^{(2)}}W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+ikW_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=o(1/r^{(2)}) \quad \text{for } r^{(\lambda)}\to\infty, \quad (7)$$

$$W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)\sim\frac{1}{r^{(\lambda)}} \quad \text{for } r^{(\lambda)}\to\infty. \quad (8)$$

The cross-spectral tensor also obeys the constraint

$$W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=W_{ji}^*(\mathbf{r}^{(2)},\mathbf{r}^{(1)},\omega). \quad (9)$$

Here $\partial_m^{(\lambda)}\equiv\partial/\partial r_m^{(\lambda)}$, $m=1,2,3$, $r^{(\lambda)}\equiv|\mathbf{r}^{(\lambda)}|$, $\lambda=1,2$, and $k=\omega/c$ with c being a speed of light *in vacuo*. Repetition of the same index (denoting a component of vector or tensor) implies summation.

Equation (5) is a straightforward consequence of the wave equation. Equations (6) are analogous to Eq. (1) and Eqs. (7) are analogous to Eq. (2). Property (9) follows directly from the definition [see Eqs. (3) and (4)]. The second equations in the pairs (6) and (7) are consequences of the first equations and Eq. (9).

III. SOLUTION OF EQUATIONS FOR THE ELECTRIC CROSS-SPECTRAL TENSOR

Suppose that $\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$ is an arbitrary solution of the equations

$$\partial_m^{(\lambda)}\partial_m^{(\lambda)}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+k^2\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=0, \quad (10)$$

$$\frac{\partial}{\partial r^{(1)}}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)-ik\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=o(1/r^{(1)}),$$
$$\frac{\partial}{\partial r^{(2)}}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+ik\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=o(1/r^{(2)}) \quad \text{for } r^{(\lambda)}\to\infty, \quad (11)$$

$$\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)\sim\frac{1}{r^{(\lambda)}} \quad \text{for } r^{(\lambda)}\to\infty, \quad (12)$$

$$\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\Phi^*(\mathbf{r}^{(2)},\mathbf{r}^{(1)},\omega) \quad (13)$$

($\lambda=1,2$). Then the appropriate solution of Eqs. (5)–(9) can be written in the form

$$W_{ij}^{(\Phi)}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\alpha(\omega)\,\mathfrak{M}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+\beta(\omega)\,\mathfrak{N}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+\gamma(\omega)\,\mathfrak{K}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)+\gamma^*(\omega)\,\mathfrak{L}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega), \quad (14)$$

where

$$\mathfrak{M}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\epsilon_{ikl}\epsilon_{jmn}\partial_k^{(1)}\partial_m^{(2)}[r_l^{(1)}r_n^{(2)}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)],$$
$$\mathfrak{N}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\frac{1}{k^2}\epsilon_{ikl}\epsilon_{jop}\epsilon_{lmn}\epsilon_{pqs}\partial_k^{(1)}\partial_o^{(2)}\partial_m^{(1)}\partial_q^{(2)}[r_n^{(1)}r_s^{(2)}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)],$$
$$\mathfrak{K}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\frac{1}{k}\epsilon_{ikl}\epsilon_{jmn}\epsilon_{nop}\partial_k^{(1)}\partial_m^{(2)}\partial_o^{(2)}[r_l^{(1)}r_p^{(2)}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)],$$
$$\mathfrak{L}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\frac{1}{k}\epsilon_{ikl}\epsilon_{lmn}\epsilon_{jop}\partial_k^{(1)}\partial_m^{(1)}\partial_o^{(2)}[r_n^{(1)}r_p^{(2)}\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)], \quad (15)$$

$\gamma(\omega)$ is an arbitrary complex function of frequency, and $\alpha(\omega),\beta(\omega)$ are any real functions of frequency. Expressions (15) can be further simplified using the identity $\epsilon_{ijk}\epsilon_{klm}=\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}$. The symbol ϵ_{ijk} is the Levi-Civita unit antisymmetric tensor and δ_{ij} is the Kronecker unit tensor. The last two tensor functions in Eqs. (15) are related as follows: $\mathfrak{K}_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\mathfrak{L}_{ji}^*(\mathbf{r}^{(2)},\mathbf{r}^{(1)},\omega)$. The solution (14) with tensor functions (15) is built in an analogous way as the solution of "single" vector Helmholtz equation (see, e.g., [22]).

Since any solution of Eqs. (10)–(13) can be expressed as an expansion using spherical harmonic functions (see, e.g., [13])

$$\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)=\sum_{n_1=0}^{\infty}\sum_{l_1=-n_1}^{n_1}\sum_{n_2=0}^{\infty}\sum_{l_2=-n_2}^{n_2}\phi_{n_1,l_1,n_2,l_2}(\omega)h_{n_1}^*(kr^{(1)})h_{n_2}(kr^{(2)})\,Y_{n_1,l_1}^*(\Omega^{(1)})Y_{n_2,l_2}(\Omega^{(2)}), \quad (16)$$

where $Y_{n,j}(\Omega)$ are the normalized spherical harmonics and $h_n(x)$ are the spherical Hankel functions of the second kind, every solution of Eqs. (5)–(9) can be written as a linear combination of products of the vector functions $\mathbf{M}^{[n,l]}(\mathbf{r},\omega)$ and $\mathbf{N}^{[n,l]}(\mathbf{r},\omega)$:

$$W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega) = \sum_{n_1=1}^{\infty}\sum_{l_1=-n_1}^{n_1}\sum_{n_2=1}^{\infty}\sum_{l_2=-n_2}^{n_2}\Big\{a_{n_1,l_1,n_2,l_2}(\omega)\,[M_i^{[n_1,l_1]}(\mathbf{r}^{(1)},\omega)]^*M_j^{[n_2,l_2]}(\mathbf{r}^{(2)},\omega) + b_{n_1,l_1,n_2,l_2}(\omega)$$
$$\times[N_i^{[n_1,l_1]}(\mathbf{r}^{(1)},\omega)]^*N_j^{[n_2,l_2]}(\mathbf{r}^{(2)},\omega) + c_{n_1,l_1,n_2,l_2}(\omega)\,[M_i^{[n_1,l_1]}(\mathbf{r}^{(1)},\omega)]^*N_j^{[n_2,l_2]}(\mathbf{r}^{(2)},\omega)$$
$$+d_{n_1,l_1,n_2,l_2}(\omega)\,[N_i^{[n_1,l_1]}(\mathbf{r}^{(1)},\omega)]^*M_j^{[n_2,l_2]}(\mathbf{r}^{(2)},\omega)\Big\}. \quad (17)$$

In Eq. (16) spherical coordinates are used and $\Omega \equiv (\vartheta,\varphi)$. Definitions of the functions $\mathbf{M}^{[n,l]}(\mathbf{r},\omega)$, $\mathbf{N}^{[n,l]}(\mathbf{r},\omega)$, and some orthogonality equations are given in the Appendix. From Eq. (9) it directly follows that

$$\begin{aligned} a_{n_1,l_1,n_2,l_2}(\omega) &= a^*_{n_2,l_2,n_1,l_1}(\omega),\\ b_{n_1,l_1,n_2,l_2}(\omega) &= b^*_{n_2,l_2,n_1,l_1}(\omega),\\ c_{n_1,l_1,n_2,l_2}(\omega) &= d^*_{n_2,l_2,n_1,l_1}(\omega). \end{aligned} \quad (18)$$

The first two relations imply that $a_{n,l,n,l}(\omega)$ and $b_{n,l,n,l}(\omega)$ are real. From the well-known non-negative definiteness of cross-spectral tensor, i.e.,

$$\int_{4\pi}\int_{4\pi} W_{ij}f_i^*(\Omega^{(1)})f_j(\Omega^{(2)})d\Omega^{(1)}d\Omega^{(2)} \geq 0 \quad (19)$$

for any vector function $\mathbf{f}(\Omega)$ with square-integrable components and for arbitrary $r^{(1)},r^{(2)}$, it follows that $a_{n,l,n,l}(\omega)$ and $b_{n,l,n,l}(\omega) \geq 0$. The constraint (19) can be derived in a manner similar to that given in [23], starting from the obvious inequality $\langle|\int_{4\pi}\int_{-\infty}^{\infty} f_i(\Omega)g(t)E_i(\mathbf{r},t)dt\,d\Omega|^2\rangle \geq 0$ [$g(t)$ is any square-integrable function of time].

IV. SPHERICALLY SYMMETRIC SOLUTION

Under terms spherically symmetric or rotationally invariant field we mean the stochastic field which "looks the same" from any direction of observation. It means the statistical characteristics of the field (at least the cross-spectral tensor of the second order) do not change their functional form under rotations of coordinates. More explicitly, if A is a transformation matrix corresponding to a rotation of the coordinate system ($\mathbf{r}' = \mathsf{A}\cdot\mathbf{r}$ or in components $r_i' = A_{ij}r_j$) then

$$A_{il}A_{jm}W_{lm}(\mathsf{A}^{-1}\cdot\mathbf{r}^{(1)},\mathsf{A}^{-1}\cdot\mathbf{r}^{(2)},\omega) = W_{ij}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega). \quad (20)$$

A deterministic spherically symmetric radiative electromagnetic field does not exist. This is a consequence of the transversity of electromagnetic waves. [And this is connected with the fact that the vector functions $\mathbf{M}^{[0,0]}(\mathbf{r},\omega)$, $\mathbf{N}^{[0,0]}(\mathbf{r},\omega)$ corresponding to the only rotationally invariant spherical harmonic function $Y_{0,0}(\Omega)$ are zero; see the Appendix.]

Nevertheless, the spherically symmetric solution of Eqs. (5)–(9) for cross-spectral tensor describing stochastic field exists. If the solution of Eqs. (10)–(13), $\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$, is spherically symmetric (as a function of two spatial variables) then the cross-spectral tensor $W_{ij}^{(\Phi)}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$ [see Eq. (14)] is also spherically symmetric, in the sense of Eq. (20). This follows from Eqs. (14) and (15) and from transformation properties of $\epsilon_{ijk},\delta_{ij},\partial_i$, and r_i. The existence of rotationally invariant $\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$ can be demonstrated on a simple example (in more special forms given in [13,14]):

$$\Phi(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega) = p(\omega)\,h_0^*(kr^{(1)})h_0(kr^{(2)})\,Y_{0,0}^*(\Omega^{(1)})Y_{0,0}(\Omega^{(2)}) + q(\omega)\,h_1^*(kr^{(1)})h_1(kr^{(2)})$$
$$\times\left[Y_{1,-1}^*(\Omega^{(1)})Y_{1,-1}(\Omega^{(2)}) + Y_{1,0}^*(\Omega^{(1)})Y_{1,0}(\Omega^{(2)}) + Y_{1,1}^*(\Omega^{(1)})Y_{1,1}(\Omega^{(2)})\right], \quad (21)$$

where $p(\omega),q(\omega)$ are any non-negative real functions of frequency. This function is evidently rotationally invariant because $Y_{0,0}(\Omega) = 1/\sqrt{4\pi}$ and

$$\sum_{m=-n}^{n} Y_{n,m}^*(\Omega^{(1)})Y_{n,m}(\Omega^{(2)}) = \frac{2n+1}{4\pi}P_n(\cos\chi), \quad (22)$$

where $P_n(x)$ is the Legendre polynomial and $\cos\chi = \mathbf{u}^{(1)}\cdot\mathbf{u}^{(2)} = \cos\vartheta^{(1)}\cos\vartheta^{(2)} + \sin\vartheta^{(1)}\sin\vartheta^{(2)}\cos(\varphi^{(1)}-\varphi^{(2)})$ (see, e.g., [24]) depends only on the angle χ between unit vectors $\mathbf{u}^{(1)},\mathbf{u}^{(2)}$ given by $\Omega^{(1)},\Omega^{(2)}$, respectively.

In general, each spherically symmetric scalar function of two spatial variables can be written as a superposition of the Legendre polynomials of arguments $\mathbf{u}^{(1)}\cdot\mathbf{u}^{(2)}$, so that for such functions the coefficients in the expansion (16) have the form $\phi_{n_1,l_1,n_2,l_2}(\omega) = \hat{\phi}_{n_1}(\omega)\delta_{n_1,n_2}\delta_{l_1,l_2}$.

V. SPECTRUM OF RADIATION

According to the well-known Wiener-Khintchine theorem, the spectrum of electric field may be defined by the expression

$$S(\mathbf{r},\omega)= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle \mathbf{E}^*(\mathbf{r},t)\cdot\mathbf{E}(\mathbf{r},t+\tau)\rangle\, e^{-i\omega\tau}d\tau = W_{ii}(\mathbf{r},\mathbf{r},\omega), \quad (23)$$

where summation over repeated indices is to be understood. The radiation field is, of course, assumed statistically stationary.

We will now derive an expression for the spectrum of a spherically symmetric field. In this case the spectrum cannot depend on direction—it may be only a function of radial distance r. Thus, by substituting from Eq. (17) into Eq. (23), integrating over 4π solid angle [the left-hand side of Eq. (23) is only multiplied by 4π] and using orthogonality of the set of functions $\mathbf{M}^{[n,l]}(\mathbf{r},\omega)$, $\mathbf{N}^{[n,l]}(\mathbf{r},\omega)$ (see the Appendix) we finally obtain for the spectrum the expression

$$S(r,\omega) = \frac{1}{4\pi}\,\frac{1}{k^2r^2}\sum_{n=1}^{\infty}\sum_{l=-n}^{n} n(n+1)\left(a_{n,l,n,l}(\omega)\,\eta_n(kr) + b_{n,l,n,l}(\omega)\,\frac{1}{2n+1}\left[(n+1)\,\eta_{n-1}(kr) + n\,\eta_{n+1}(kr)\right]\right), \quad (24)$$

where

$$\eta_n(x) = x^2|h_n(x)|^2 = \sum_{j=0}^{n}\frac{(2n-j)!(2n-2j)!}{j![(n-j)!]^2}(2x)^{2j-2n} \quad (25)$$

(see [25]). Equation (24) can be rewritten in the form

$$S(r,\omega) = \frac{1}{4\pi}\,\frac{1}{r^2}\sum_{n=0}^{\infty}\psi_n(\omega)\,\eta_n(kr), \quad (26)$$

where

$$\psi_n(\omega) = \frac{1}{k^2}\left[\sum_{l=-n}^{n} a_{n,l,n,l}(\omega)\,n(n+1) + \sum_{l=-n+1}^{n-1} b_{n-1,l,n-1,l}(\omega)\frac{(n-1)^2 n}{2n-1} + \sum_{l=-n-1}^{n+1} b_{n+1,l,n+1,l}(\omega)\frac{(n+1)(n+2)^2}{2n+3}\right]. \quad (27)$$

Of course, for $n = 0$ the first and the second terms are zero and for $n = 1$ the second term is zero. Because of the non-negativeness of $a_{n,l,n,l}(\omega)$ and $b_{n,l,n,l}(\omega)$, the coefficients $\psi_n(\omega)$ are also non-negative.

Equation (26) allows us to determine the spectrum at any spatial point from the knowledge of the coefficients $\psi_n(\omega)$. These coefficients can be evaluated from the knowledge of the cross-spectral tensor on some sphere (this represents a boundary condition from which one can determine completely the cross-spectral tensor everywhere outside the sphere; the sphere can be regarded as a surface of a source).

The frequency dependencies of $\psi_n(\omega)$ are modified (for each fixed r) by factors $\eta_n(kr)$ and for fixed frequency the modes with higher n decrease faster with increasing r than those of smaller n (this fact is connected with the improvement of coherent properties of radiation on propagation). Because of this, the functional dependence of the spectrum on the frequency may be different at different distances from the origin. The spectrum normalized with respect to maximum value (in the frequency domain) can change its functional form from point to point.

The coefficients $\psi_n(\omega)$ corresponding to the simple example built on the basis of the scalar function given by Eq. (21) are $\psi_0(\omega) = 4\beta(\omega)q(\omega)$, $\psi_1(\omega) = 6\alpha(\omega)q(\omega)$, and $\psi_2(\omega) = 2\beta(\omega)q(\omega)$ (the others are zero). Here $q(\omega)$ originates from Eq. (21) and $\alpha(\omega)$, $\beta(\omega)$ are coefficients from Eq. (14) (but now they are non-negative because of the non-negative definiteness of cross-spectral tensor).

The form of Eq. (26), which is rather similar to the form of equation for the spectrum obtained within the domain of scalar theory, is not surprising, because from Eqs. (5) and (7)–(9) it is clear that $W_{ii}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$ (summed over i) have to satisfy Eqs. (10)–(13) [which are equivalent to Eqs. (3)–(5) in Ref. [13]]. Nevertheless, there is an important difference. The set of possible sequences of coefficients $\psi_n(\omega)$ in Eq. (26) is restricted by Eqs. (6). Above all, if $\psi_0(\omega) \neq 0$ for some ω, then at least the coefficient $\psi_2(\omega)$ has to be also nonzero as can be seen from Eq. (27). It means that the spectrum cannot depend on the radial distance r in a trivial manner, i.e., through factor $1/r^2$ $[\eta_0(kr) = 1]$. It even seems that the spectrum cannot be factorized, i.e., expressed as the product of a function of frequency and a function of radius, though this result has not yet been proved.

The spectrum of magnetic field can be obtained by replacing mutually the coefficients $a_{n,l,n,l}(\omega)$ and $b_{n,l,n,l}(\omega)$ in the expressions for the spectrum of electric field and, of course, by multiplying these expressions by a constant depending on the units used [26].

VI. ENERGY CONSERVATION

From the conservation law for deterministic fields and from the definition of the operation of averaging it is clear that energy has to be conserved also for any stochastic electromagnetic field; see, e.g., [17,27]. As an example, showing at once that our foregoing calculations are correct, we derive an explicit form of the law of conservation of energy for the field described before.

The average energy flux is defined as

$$\mathbf{F}(\mathbf{r},t) = \kappa \,\mathrm{Re}\,\langle \mathbf{E}^*(\mathbf{r},t) \times \mathbf{H}(\mathbf{r},t)\rangle\,, \tag{28}$$

where the complex vector $\mathbf{H}(\mathbf{r},t)$ is constructed by the same manner as $\mathbf{E}(\mathbf{r},t)$, but on the basis of the magnetic field vector, and κ is a real constant depending on the choice of units. Re denotes the real part.

Statistical average of the conservation law has the form

$$\nabla \cdot \mathbf{F}(\mathbf{r},t) = 0 \tag{29}$$

in any region which does not contain sources. The time variation of the averaged energy density is zero in the case of a statistically stationary field.

Let us introduce the vector

$$\boldsymbol{\mathcal{F}}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},t^{(2)}-t^{(1)}) = \kappa \left\langle \mathbf{E}^*(\mathbf{r}^{(1)},t^{(1)}) \times \mathbf{H}(\mathbf{r}^{(2)},t^{(2)})\right\rangle . \tag{30}$$

From Maxwell's equations it follows that

$$\frac{\partial}{\partial t^{(2)}}\mathcal{F}_i(\mathbf{r}^{(1)},\mathbf{r}^{(2)},t^{(2)}-t^{(1)}) = \widehat{\kappa}\,\epsilon_{ijk}\epsilon_{klm}\partial_l^{(2)}\left\langle E_j^*(\mathbf{r}^{(1)},t^{(1)})E_m(\mathbf{r}^{(2)},t^{(2)})\right\rangle , \tag{31}$$

where $\widehat{\kappa}$ is a real constant. Performing the Fourier transform of Eq. (31) with respect to $\tau = t^{(2)} - t^{(1)}$ we obtain the equation

$$\widetilde{\mathcal{F}}_i(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega) = \frac{\widehat{\kappa}}{i\omega}\epsilon_{ijk}\epsilon_{klm}\partial_l^{(2)}W_{jm}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega), \tag{32}$$

where

$$\widetilde{\mathcal{F}}_i(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathcal{F}_i(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\tau)e^{-i\omega\tau}d\tau. \tag{33}$$

Substituting from Eq. (17) into Eq. (32) and using the fact that

$$k\,\mathbf{M}^{[n,l]}(\mathbf{r},\omega) = \nabla \times \mathbf{N}^{[n,l]}(\mathbf{r},\omega),$$

$$k\,\mathbf{N}^{[n,l]}(\mathbf{r},\omega) = \nabla \times \mathbf{M}^{[n,l]}(\mathbf{r},\omega)$$

[see Eqs. (A2)] we could write the explicit form of vector $\widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r}^{(1)},\mathbf{r}^{(2)},\omega)$. Then it is easy to derive $\nabla \cdot \widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r},\mathbf{r},\omega)$ (the operator nabla is acting with respect to variable $\mathbf{r}$). Using the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

and employing again Eqs. (A2) we obtain a series for $\nabla \cdot \widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r},\mathbf{r},\omega)$, which contains the scalar products of type

$$[\mathbf{M}^{[n_1,l_1]}(\mathbf{r},\omega)]^* \cdot \mathbf{M}^{[n_2,l_2]}(\mathbf{r},\omega),$$

$$[\mathbf{M}^{[n_1,l_1]}(\mathbf{r},\omega)]^* \cdot \mathbf{N}^{[n_2,l_2]}(\mathbf{r},\omega),$$

etc. Reordering the series and using relations (18) one can readily show that $\nabla \cdot \widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r},\mathbf{r},\omega)$ is purely imaginary. Thus

$$\mathrm{Re}[\nabla \cdot \widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r},\mathbf{r},\omega)] = 0 \tag{34}$$

and

$$\nabla \cdot \mathbf{F}(\mathbf{r},t) = \nabla \cdot \mathrm{Re}\int_0^{\infty}\widetilde{\boldsymbol{\mathcal{F}}}(\mathbf{r},\mathbf{r},\omega)d\omega = 0. \tag{35}$$

These formulas show that energy is conserved at each frequency.

We note that the spherical symmetry have not been used in this consideration.

VII. CONCLUSIONS

We have described spherically symmetric statistically stationary stochastic electromagnetic field within the framework of the second-order correlation theory. We have noted that the spectrum is, in general, "non-invariant" on propagation and have shown by explicit calculations that there is no contradiction with the energy conservation law.

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APPENDIX: VECTOR SPHERICAL WAVE FUNCTIONS

The solution of the equation

$$\nabla^2\mathbf{C} + k^2\mathbf{C} = \mathbf{0}, \tag{A1}$$

where $\nabla^2\mathbf{C} = \nabla\nabla\cdot\mathbf{C} - \nabla\times\nabla\times\mathbf{C}$, can be written (see [22]) as a superposition of the functions [28]

$$\mathbf{L}^{[n,j]}(\mathbf{r},\omega)= \nabla\,[Y_{n,j}(\Omega)z_n(kr)]\,,$$
$$\mathbf{M}^{[n,j]}(\mathbf{r},\omega)= \mathbf{L}^{[n,j]}(\mathbf{r},\omega)\times\mathbf{r} = \frac{1}{k}\nabla\times\mathbf{N}^{[n,j]}(\mathbf{r},\omega)$$
$$= \nabla\times[\mathbf{r}Y_{n,j}(\Omega)z_n(kr)]\,,$$
$$\mathbf{N}^{[n,j]}(\mathbf{r},\omega)= \frac{1}{k}\nabla\times\mathbf{M}^{[n,j]}(\mathbf{r},\omega)$$
$$= \frac{1}{k}\nabla\times\{\nabla\times[\mathbf{r}Y_{n,j}(\Omega)z_n(kr)]\},\qquad \text{(A2)}$$

where $z_n(x)$ denote suitable radial functions. In this paper we identify $z_n(x)$ with the spherical Hankel functions of the second kind because expression (17) then satisfies the radiation conditions (7). Only the functions $\mathbf{M}^{[n,j]}(\mathbf{r},\omega)$, $\mathbf{N}^{[n,j]}(\mathbf{r},\omega)$ have zero divergences and consequently only they are convenient for description of electromagnetic radiation fields. From Eqs. (A2) it follows that $\mathbf{M}^{[0,0]}(\mathbf{r},\omega) = \mathbf{N}^{[0,0]}(\mathbf{r},\omega) = \mathbf{0}$.

The orthogonality relations for functions $\mathbf{M}^{[n,j]}(\mathbf{r},\omega)$, $\mathbf{N}^{[n,j]}(\mathbf{r},\omega)$ $[z_n(x) = h_n(x)]$:

$$\int_{4\pi}[\mathbf{M}^{[n,j]}(\mathbf{r},\omega)]^*\cdot\mathbf{M}^{[n',j']}(\mathbf{r},\omega)d\Omega = \delta_{n,n'}\,\delta_{j,j'}\,n(n+1)\,|h_n(kr)|^2,$$
$$\int_{4\pi}[\mathbf{N}^{[n,j]}(\mathbf{r},\omega)]^*\cdot\mathbf{N}^{[n',j']}(\mathbf{r},\omega)d\Omega = \delta_{n,n'}\,\delta_{j,j'}\,\frac{n(n+1)}{2n+1}\{(n+1)\,|h_{n-1}(kr)|^2 + n\,|h_{n+1}(kr)|^2\}.\qquad \text{(A3)}$$

All other combinations give zero.

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